

Poisson-Sigma Models*

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Abstract

We investigate the Poisson-Sigma model on the classical and quantum level. In the classical analysis we show how this model includes various known two-dimensional field theories. Then we perform the calculation of the path integral in a general gauge, and demonstrate that the derived partition function reduces to the familiar form in the case of 2d Yang-Mills theory.

1 Introduction

The Poisson-Sigma model [1] is a gauge theory based on a Poisson algebra, i.e. it is a non-linear extension of an ordinary gauge theory, which is based on a linear Lie algebra. The class of these models, which are based on a non-linear Lie algebra, i.e. a finite W-algebra or a Poisson algebra, are known in this context as *non-linear gauge theory* [2]. The Poisson-Sigma model associates to any Poisson structure on a finite-dimensional manifold a two-dimensional field theory [3]. Choosing different Poisson structures leads to specific models which include most of the topological and semi-topological field theories which have been of interest in recent years. These include gravity models, non-abelian gauge theories and the Wess-Zumino-Witten model.

Because of the non-linearity of the algebra the Poisson-Sigma model involves in the language of gauge theories an open gauge algebra, i.e. the algebra closes only on-shell. In such cases the Faddeev-Popov method of path integral quantization fails. Quantization procedures which rely on the BRST symmetry of the extended action are in principle more powerful [4]. We find that the antifield formalism of Batalin and Vilkovisky [5] is the most effective method to get a suitable action for the path integral quantization. The path integral for the Poisson-Sigma model was first discussed in a preliminary way by Schaller and Strobl in [3]. In a recent paper Cattaneo and Felder [6] used the perturbation expansion of the path integral in the covariant gauge to elucidate Kontsevich's formula for the deformation quantization of the algebra of functions on a Poisson manifold [7]. Kummer et.al. have investigated the special case of 2d dilaton gravity and they have calculated the generating functional using BRST methods [8]. We have investigated in a recent paper the path integral quantization of the Poisson-Sigma model in a general gauge and derived an almost closed expression for the partition function [9].

The article is structured as follows. In Section 2 we introduce the Poisson-Sigma model and show how the model reduces to known field theories. In Section 3 we construct the Batalin-Vilkovisky action and perform the calculation of the path integral. We also show how the derived partition function for the general model reduces under certain circumstances to the more familiar Yang-Mills case [10]. Section 4 contains some concluding remarks.

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2 The Poisson-Sigma Model

2.1 Poisson manifolds and the model

A *Poisson manifold* (N, P) is a smooth manifold N equipped with a Poisson structure $P \in \Lambda^2 TN$ [11]. In local coordinates X^i on N

$$P = \frac{1}{2} P^{ij}(X) \partial_i \wedge \partial_j, \quad (2.1)$$

and P^{ij} has to satisfy the condition

$$P^{i[j} P^{lk]}_{,i} = 0, \quad (2.2)$$

which reflects the vanishing of the corresponding Schouten-Nijenhuis bracket for P with itself. Here the bracketed indices denote an antisymmetric sum. In the notation of Poisson brackets

$$\{f(X), g(X)\} = P^{ij}(X) f_{,i} g_{,j} \quad (2.3)$$

and the Jacobi identity follows from Eq. (2.2):

$$\{f, \{g, h\}\} + \text{cyclic} = 0. \quad (2.4)$$

The Poisson bracket satisfies the Leibniz derivation rule:

$$\{h, fg\} = \{h, f\}g + f\{h, g\}, \quad (2.5)$$

where f, g, h are functions on the manifold N . These facts are of an algebraic nature, and it is natural to define a *Poisson algebra* as an associative commutative algebra endowed with a bracket that satisfies (2.3), (2.4) and (2.5). Indeed, a smooth manifold N is called a Poisson manifold if the algebra of smooth functions is a Poisson algebra.

The splitting theorem of Weinstein [12] states that for a regular Poisson manifold, i.e. the Poisson tensor has constant rank, there exist so-called Casimir-Darboux coordinates on the Poisson manifold (N, P) . For P degenerate there are nonvanishing functions f on N whose Hamiltonian vector fields $X_f = f_{,i} P^{ij} \partial_j$ vanish. These functions are called Casimir functions. Let $\{C^I\}$ be a maximal set of independent Casimir functions. Then $C^I(X) = \text{const.} = C^I(X_0)$ defines a level surface through X_0 whose connected components may be identified with the symplectic leaves S which constitute the symplectic foliation of the Poisson manifold (N, P) . According to Darboux's theorem there are local coordinates X^α on S such that the symplectic form Ω_S is given by

$$\Omega_S = dX^1 \wedge dX^2 + dX^3 \wedge dX^4 + \dots \quad (2.6)$$

Together with the Casimir functions we then have a natural coordinate system $\{X^I, X^\alpha\}$ on N with $P^{IJ} = P^{I\alpha} = 0$ and $P^{\alpha\beta} = \text{constant}$.

The Poisson-Sigma model is a field theory on a closed two-dimensional world sheet M . First it involves a set of bosonic scalar fields X^i , which can be interpreted as mappings from the world sheet to a Poisson manifold, $X^i : M \rightarrow N$. In addition one needs fields A_i which are one-forms on M taking values in T^*N , i.e. one-forms on the world sheet which are simultaneously the pullback of sections of T^*N by the map $X(x)$, where x denotes the coordinates of the world sheet. These fields can be seen as a non-linear extension of the gauge fields of an ordinary gauge theory. The action of the semi-topological Poisson-Sigma model is:

$$\mathcal{S}_0[X, A] = \int_M \left[A_i \wedge dX^i + \frac{1}{2} P^{ij}(X) A_i \wedge A_j + \mu C(X) \right], \quad (2.7)$$

where μ is the volume form on M , $C(X)$ is a Casimir function and d denotes an exterior derivative on the world sheet M . Note that the Casimir function in the action breaks the topological nature of the theory.

The action is invariant under the following symmetry transformations:

$$\delta X^i = P^{ij}(X)\varepsilon_j, \quad \delta A_i = D_i^j \varepsilon_j, \quad (2.8)$$

where $D_i^j = \delta_i^j d + P^{kj}{}_{,i} A_k$ is the covariant derivative on M .

The equations of motion are

$$D_i^j A_j + \frac{\partial C(X)}{\partial X^i} = 0 \quad \text{and} \quad dX^i + P^{ij} A_j = DX^i = 0. \quad (2.9)$$

The symmetry algebra is given by:

$$\begin{aligned} [\delta(\varepsilon_1), \delta(\varepsilon_2)]X^i &= P^{ji}(P^{mn}{}_{,j} \varepsilon_{1n} \varepsilon_{2m}), \\ [\delta(\varepsilon_1), \delta(\varepsilon_2)]A_i &= D_i^j (P^{mn}{}_{,j} \varepsilon_{1n} \varepsilon_{2m}) - (DX^j) P^{mn}{}_{,ji} \varepsilon_{1n} \varepsilon_{2m}. \end{aligned} \quad (2.10)$$

Note that here the non-linearity of the Poisson algebra is manifested in the additional term which is proportional to the variation of the action with respect to A_i .

2.2 Three Examples

Now we want to show how the Poisson-Sigma model reduces to specific two-dimensional field theories. This goal can be achieved by the choice of a particular Poisson structure on the Poisson manifold, which corresponds to a choice of the target manifold.

Non-degenerate Poisson structure: the first example concerns the case of the nondegenerate Poisson structure, i.e the Poisson structure P has an inverse Ω . This is exactly the symplectic 2-Form on the manifold N . It turns out that the target space is now a symplectic manifold. Note that in this case the only Casimir function is the trivial function $C = 0$. It is possible to solve the equations of motion for the fields A_i , and one has:

$$A_i = \Omega_{ij} dX^j. \quad (2.11)$$

Using this equation to eliminate the fields A_i in the action, one gets:

$$\mathcal{S}_{top} = \int_M \Omega_{ij} dX^i \wedge dX^j. \quad (2.12)$$

This is exactly the action of the *topological Sigma model* proposed by E.Witten [13] in the Baulieu-Singer approach [14].

Linear Poisson structure: next we consider a linear Poisson structure $P^{ij} = c_k^{ij} X^k$ on the three dimensional space \mathbb{R}^3 . Because of the Jacobi identity the structure coefficients c_k^{ij} define a *Lie algebra* structure on the dual space \mathcal{G} , of N . For this reason the linear Poisson structure is also called a Lie-Poisson structure on N . It is not hard to see that the fields A_i reduce to the ordinary gauge fields and their covariant derivative is now the ordinary curvature of a gauge theory:

$$F_i = D_i^j A_j = dA_i + \frac{1}{2} c_i^{kl} A_k \wedge A_l. \quad (2.13)$$

For a linear Poisson structure there exist two different types of Casimir functions, namely the trivial case $C = 0$ and the quadratic Casimir $C = \sum_i X^i X^i$.

For $C = 0$ the action is given by:

$$\mathcal{S}_{BF} = \int_M X^i F_i, \quad (2.14)$$

and one sees that it is the action of a *topological BF gauge theory* [15].

Choosing now the quadratic Casimir yields for the action after a short calculation:

$$\mathcal{S}_{YM} = \int_M F^i \wedge *F_i, \quad (2.15)$$

where $*$ denotes the Hodge Star operator on the world sheet M . This is now the action of a *2d Yang-Mills theory*.

2-dimensional Gravity: as a last example we want to see what happens if we choose a non-linear Poisson structure on \mathbb{R}^3 given by:

$$P^{ij} = \epsilon^{ijk} u_k(X), \quad (2.16)$$

$$\text{with } u_a = \eta_{ab} X^b, \text{ where } a, b = 1, 2 \quad (2.17)$$

$$\text{and } u_3 = V(X^a \eta_{ab} X^b, X^3). \quad (2.18)$$

Choosing the trivial Casimir function $C = 0$ the action is:

$$\mathcal{S} = \int_M \left(X^a \mathcal{D}_3 A_a + X^3 dA_3 + \frac{1}{2} V \epsilon^{ab} A_a \wedge A_b \right). \quad (2.19)$$

If we now identify the first two components of the field A_i with the *Zweibein* e_a and the third with the *spin connection* ω we get for the action:

$$\mathcal{S} = \int_M \left(X^a \mathcal{D}_\omega e_a + X^3 d\omega + \frac{1}{2} V \epsilon^{ab} e_a \wedge e_b \right) \quad (2.20)$$

Hence we get *two-dimensional dilaton gravity* with a potential V depending on the dilaton field X^3 [16].

3 Quantization of the Model

3.1 The Batalin-Vilkovisky Action

We use the path integral approach for quantization, because we are interested in the quantum theory for different space-time topologies. To obtain a suitable action for the path integral we apply the Batalin-Vilkovisky method for the Poisson-Sigma Model. We just point out some of the essential ingredients of this approach, for a general description see for example [17].

The first thing one has to do is to introduce ghosts C for the symmetry transformations, and for each field Φ^A a corresponding antifield Φ_A^* . For the Poisson-Sigma model one has:

$$\Phi^A = \{A_{\mu i}, X^i, C_i\} \text{ and } \Phi_A^* = \{A^{\mu i*}, X_i^*, C^{i*}\}. \quad (3.1)$$

For the fields and antifields there are two gradings, one is the form-degree with respect to the world sheet M and the other is the *ghost number*:

gh \ deg	0	1	2
-2			C^{i*}
-1		A^{i*}	X_i^*
0	X^i	A_i	
1	C_i		

On the space of the fields and antifields one defines a symplectic structure using the *antibracket*. For the fields B and C this bracket is given by:

$$(B, C) = \sum_A \left[\frac{\partial B}{\partial \Phi^A} \wedge \frac{\partial C}{\partial \Phi_A^*} - (-1)^{\deg(\Phi^A)} \frac{\partial C}{\partial \Phi_A^*} \wedge \frac{\partial B}{\partial \Phi^A} \right]. \quad (3.2)$$

One also defines a *Laplace operator*:

$$\Delta C = \sum_A (-1)^{\text{gh}(A)} \frac{\partial^2 C}{\partial \Phi^A \partial \Phi_A^*}. \quad (3.3)$$

The required action has to fulfill the following conditions. First, for the vanishing of the antifields the original classical action should be obtained. And second, the action must satisfy the *Quantum Master Equation*:

$$(\mathcal{S}_{BV}, \mathcal{S}_{BV}) - 2\hbar i \Delta \mathcal{S}_{BV} = 0. \quad (3.4)$$

For the Poisson-Sigma model the extended action which satisfies both conditions is:

$$\begin{aligned} \mathcal{S}_{BV} = \int_M \left[A_i \wedge dX^i + \frac{1}{2} P^{ij}(X) A_i \wedge A_j + \mu C(X) + A^{i*} \wedge D_i^j C_j + X_i^* P^{ji}(X) C_j \right. \\ \left. + \frac{1}{2} C^{i*} P^{jk}{}_{,i}(X) C_j C_k + \frac{1}{4} A^{i*} \wedge A^{j*} P^{kl}{}_{,ij}(X) C_k C_l \right]. \end{aligned} \quad (3.5)$$

Transforming now the extended action into Casimir-Darboux coordinates one gets:

$$\begin{aligned} \mathcal{S}_{BV} = \int_M \left[A_I \wedge dX^I + A_\alpha \wedge dX^\alpha + \frac{1}{2} P^{\alpha\beta} A_\alpha \wedge A_\beta + \mu C(X^I) \right. \\ \left. + A^{I*} \wedge dC_I + A^{\alpha*} \wedge dC_\alpha + X_\alpha^* P^{\beta\alpha} C_\beta \right]. \end{aligned} \quad (3.6)$$

Note that there are two essential simplifications. First, the terms which are quadratic in the ghosts vanish, and second the covariant derivative reduces to the normal exterior derivative. These two facts are essential for the nonperturbative calculation of the path integral.

3.2 Gauge fixing

In the Batalin-Vilkovisky approach the gauge fixing is incorporated by the *gauge fermion* Ψ in the following manner. The unphysical antifields will be eliminated by the derivative of the gauge fermion with respect to the fields:

$$\Phi_A^* = \frac{\partial \Psi}{\partial \Phi^A}. \quad (3.7)$$

The ghost number of the gauge fermion has to be (-1) , so that additional fields are necessary. The simplest choice is a so-called *trivial pair*: \bar{C}_i , $\bar{\pi}_i$ and the corresponding antifields. In the Casimir-Darboux coordinates the gauge fermion can be chosen to be:

$$\Psi = \int_M [\bar{C}^I \chi_I(A_I) + \bar{C}^\alpha \chi_\alpha(X^\alpha)]. \quad (3.8)$$

The gauge fixed action in Casimir-Darboux coordinates is given by:

$$\mathcal{S}_\Psi = \int_M \left[A_I \wedge dX^I + A_\alpha \wedge dX^\alpha + \frac{1}{2} P^{\alpha\beta} A_\alpha \wedge A_\beta + \mu C(X^I) \right. \\ \left. + \bar{C}^J \frac{\partial \chi_J(A_J)}{\partial A_I} \wedge dC_I + \bar{C}^\alpha \frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\beta} P^{\gamma\beta} C_\gamma - \bar{\pi}^I \chi_I(A_I) - \bar{\pi}^\alpha \chi_\alpha(X^\alpha) \right]. \quad (3.9)$$

Now we have arrived at an action which can be used in the path integral, since the ambiguity in the path integral which occurs because of the gauge freedom, is removed by the incorporation of the gauge fermion.

3.3 Calculation of the Path integral

The path integral for the Poisson-Sigma model in Casimir-Darboux coordinates is

$$Z = \int_{\Sigma_\Psi} \mathcal{D}X^I \mathcal{D}X^\alpha \mathcal{D}A_I \mathcal{D}A_\alpha \mathcal{D}C_I \mathcal{D}\bar{C}_I \mathcal{D}C_\alpha \mathcal{D}\bar{C}_\alpha \mathcal{D}\bar{\pi}_I \mathcal{D}\bar{\pi}_\alpha \exp \left(-\frac{1}{\hbar} \mathcal{S}_\Psi \right), \quad (3.10)$$

where we have performed the usual Wick rotation $t = i\tau$, so that the exponent of the path integral is now real. Integrating over the ghost and antighost fields yields the Faddeev-Popov determinants:

$$Z = \int_{\Sigma_\Psi} \mathcal{D}X^I \mathcal{D}X^\alpha \mathcal{D}A_I \mathcal{D}A_\alpha \mathcal{D}\bar{\pi}_I \mathcal{D}\bar{\pi}_\alpha \det \left(\frac{\partial \chi_I(A_I)}{\partial A_I} \wedge d \right)_{\Omega^0(M)} \det \left(\frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X^I) \right)_{\Omega^0(M)} \\ \times \exp \left(-\frac{1}{\hbar} \int_M \left[A_I \wedge dX^I + A_\alpha \wedge dX^\alpha + \frac{1}{2} P^{\alpha\beta} A_\alpha \wedge A_\beta + \mu C(X^I) - \bar{\pi}^I \chi_I(A) - \bar{\pi}^\alpha \chi_\alpha(X^\alpha) \right] \right), \quad (3.11)$$

where the subscripts $\Omega^k(M)$ indicate that the determinant results from an integration over k -forms on M . The integrations over $\bar{\pi}_I$ and $\bar{\pi}_\alpha$ yield δ - functions which implement the gauge conditions.

$$Z = \int_{\Sigma_\Psi} \mathcal{D}X^I \mathcal{D}X^\alpha \mathcal{D}A_I \mathcal{D}A_\alpha \det \left(\frac{\partial \chi_I(A_I)}{\partial A_I} \wedge d \right)_{\Omega^0(M)} \det \left(\frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X^I) \right)_{\Omega^0(M)} \\ \times \exp \left(-\frac{1}{\hbar} \int_M \left[A_I \wedge dX^I + A_\alpha \wedge dX^\alpha + \frac{1}{2} P^{\alpha\beta} A_\alpha \wedge A_\beta + \mu C(X^I) \right] \right), \quad (3.12)$$

where from now on the integrations extend only over the degrees of freedom which respect the gauge-fixing conditions. The integration over $A_{\mu\alpha}$ is gaussian, it yields

$$Z = \int_{\Sigma_\Psi} \mathcal{D}X^I \mathcal{D}X^\alpha \mathcal{D}A_I \det \left(\frac{\partial \chi_I(A_I)}{\partial A_I} \wedge d \right)_{\Omega^0(M)} \det \left(\frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X^I) \right)_{\Omega^0(M)} \\ \times \det^{-1/2} (P^{\alpha\beta}(X^I))_{\Omega^1(M)} \exp \left(-\frac{1}{\hbar} \int_M \left[A_I \wedge dX^I + \Omega_{\alpha\beta} dX^\alpha \wedge dX^\beta + \mu C(X^I) \right] \right). \quad (3.13)$$

Besides the term in the exponent the only dependence on A_I is in the relevant Faddeev-Popov determinant. If we choose a gauge condition linear in A_I this determinant becomes independent of the fields, and can be absorbed into a normalization factor. The integration over A_I then yields a δ -function for dX^I . When this δ -function is implemented the fields X^I become independent of

the coordinates $\{x\}$ on M . Hence the Casimir functions are constants. The constant modes of the Casimir coordinates X_0^I count the symplectic leaves. The path integral is now

$$Z = \int_{\Sigma_\Psi} dX_0^I \mathcal{D}X^\alpha \det \left(\frac{\partial \chi_\alpha(X^\alpha)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right)_{\Omega^0(M)} \det^{-1/2} (P^{\alpha\beta}(X_0^I))_{\Omega^1(M)} \\ \times \exp \left(-\frac{1}{\hbar} \int_M \Omega_{\alpha\beta} dX^\alpha \wedge dX^\beta \right) \exp \left(-\int_M \frac{1}{\hbar} \mu C(X_0^I) \right). \quad (3.14)$$

The gauge-fixing of the fields X^α reduces the integral $\mathcal{D}X^\alpha$ to a sum over the homotopy classes of the maps:

$$Z = \int_{\Sigma_\Psi} dX_0^I \sum_{[M \rightarrow S(X_0^I)]} \det \left(\frac{\partial \chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right)_{\Omega^0(M)} \det^{-1/2} (P^{\alpha\beta}(X_0^I))_{\Omega^1(M)} \\ \times \exp \left(-\frac{1}{\hbar} \int_M \Omega_{\alpha\beta} dX^\alpha \wedge dX^\beta \right) \exp \left(-\frac{1}{\hbar} \int_M \mu C(X_0^I) \right). \quad (3.15)$$

Since the $C(X_0^I)$ are independent of the coordinates on M the last exponent simplifies to

$$\exp \left(-\frac{1}{\hbar} \int_M \mu C(X_0^I) \right) = \exp \left(-\frac{1}{\hbar} A_M C(X_0^I) \right), \quad (3.16)$$

where A_M is the surface area of M . The form of the path integral then becomes

$$Z = \int_{\Sigma_\Psi} dX_0^I \sum_{[M \rightarrow S(X_0^I)]} \det \left(\frac{\partial \chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right)_{\Omega^0(M)} \det^{-1/2} (P^{\alpha\beta}(X_0^I))_{\Omega^1(M)} \\ \times \exp \left(-\frac{1}{\hbar} \int_M \Omega_{\alpha\beta} dX^\alpha \wedge dX^\beta \right) \exp \left(-\frac{1}{\hbar} A_M C(X_0^I) \right). \quad (3.17)$$

Note that we have now arrived at an almost closed expression for the partition function for the Poisson-Sigma model, i.e. all the functional integrations have been performed.

3.4 The case of the linear Poisson structure

We again consider the special case where the Poisson manifold $N = \mathbb{R}^3$, and the Poisson structure is linear: $P^{ij} = c_k^{ij} X^k$. Here we are interested in the case of the quadratic Casimir function which leads to the 2d Yang-Mills theory. The corresponding symplectic leaves are two dimensional spheres characterized, in the Casimir-Darboux coordinates, by their radius X_0^I . Weinstein [12] has shown that the symplectic leaves of a linear Poisson structure are the co-adjoint orbits of the corresponding compact, connected Lie group G of \mathcal{G} . Because the Lie algebra has three dimensions we are restricted here to the case where the Lie group is the group $SU(2)$. By a theorem of Kirillov these orbits can in turn be identified with the irreducible unitary representations of G [18].

These considerations can be used to further reduce the expression for the path integral. Consider the homotopy classes of the maps $X^\alpha : M \rightarrow S(X_0^I)$. The Hopf theorem tells us that the mappings $f, g : M \rightarrow S(X_0^I)$ are homotopic if and only if the degree of the mapping f is the same as the degree of g . This means that the sum over the homotopy classes of the maps $[X^\alpha]$ can be expressed as a sum over the degrees $n = \deg[X^\alpha]$:

$$\sum_{[X^\alpha]} \rightarrow \sum_{n \in \mathbb{Z}}. \quad (3.18)$$

For a map $f : X \longrightarrow Y$, where X and Y are k -dimensional oriented manifolds and ω a k -form on Y , the degree of the mapping is given by

$$\int_X f^* \omega = \deg[f] \int_Y \omega . \quad (3.19)$$

Using this formula yields:

$$\int_M \Omega_{\alpha\beta} dX^\alpha \wedge dX^\beta = n \int_S \Omega_S(X_0^I) , \quad (3.20)$$

where $\Omega_S(X_0^I)$ is the symplectic form on the corresponding leaf S . This gives for the partition function of Eq. (3.17)

$$\begin{aligned} Z = \int_{\Sigma_\Psi} dX_0^I \sum_{n \in \mathbb{Z}} \det \left(\frac{\partial \chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right)_{\Omega^0(M)} \det^{-1/2} (P^{\alpha\beta}(X_0^I))_{\Omega^1(M)} \\ \times \exp \left(-n \int_S \Omega_S(X_0^I) \right) \exp \left(-\frac{1}{\hbar} A_M C(X_0^I) \right) . \end{aligned} \quad (3.21)$$

The sum over n yields a periodic δ -function:

$$\begin{aligned} Z = \int_{\Sigma_\Psi} dX_0^I \sum_{n \in \mathbb{Z}} \det \left(\frac{\partial \chi_\alpha(X)}{\partial X^\gamma} P^{\gamma\beta}(X_0^I) \right)_{\Omega^0(M)} \det^{-1/2} (P^{\alpha\beta}(X_0^I))_{\Omega^1(M)} \\ \times \delta \left(\int_S \Omega_S(X_0^I) - n \right) \exp \left(-\frac{1}{\hbar} A_M C(X_0^I) \right) . \end{aligned} \quad (3.22)$$

The δ -function says that the symplectic leaves must be integral. By the identification of the leaves with the co-adjoint orbits, the orbits must also be integral. The fact that the orbits are integral reduces the number of the co-adjoint orbits to a countable set, which we label by $\mathcal{O}(\Omega)$.

We now consider the two determinants in the path integral. We choose the “unitary gauge” $\chi_\alpha(X^\alpha) = X^\alpha$, so that $\partial \chi_\alpha(X)/\partial X^\gamma = \delta_\gamma^\alpha$, and the two determinants have the same form. The restriction of the scalar fields to the Casimir-Darboux coordinates X^I corresponds to the restriction of the scalar fields to the invariant Cartan subalgebra considered by Blau and Thompson in [19], so we may adopt their argumentation concerning the powers to which the determinants occur for a manifold with Euler characteristic $\chi(M)$. The result is a factor

$$\det(P^{\alpha\beta}(X_0^I))^{\chi(M)} . \quad (3.23)$$

The determinant of a mapping equals the volume of the image of that mapping, hence the determinant $\det(P^{\alpha\beta}(X_0^I))$ corresponds to the symplectic volume of the leaf, which we denote by $\text{Vol}(\Omega_S(X_0^I))$. The path integral then takes the form:

$$Z = \int_{\Sigma_\Psi} dX_0^I \sum_{n \in \mathbb{Z}} \text{Vol}(\Omega_S(X_0^I))^{\chi(M)} \delta \left(\int_S \Omega_S(X_0^I) - n \right) \exp \left(-\frac{1}{\hbar} A_M C(X_0^I) \right) . \quad (3.24)$$

Implementing the δ -function by integrating over X_0^I the sum over the mapping degrees becomes a sum over the set $\mathcal{O}(\Omega)$ of the integral orbits:

$$Z = \sum_{\mathcal{O}(\Omega)} \text{Vol}(\Omega_S(X_0^I))^{\chi(M)} \exp \left(-\frac{1}{\hbar} A_M C(X_0^I) \right) . \quad (3.25)$$

Because of the identification of the integral orbits with the irreducible unitary representations this leads to a sum over the representations. A special form of the character formula of Kirillov [20] says that the symplectic volume of the co-adjoint orbit equals the dimension of the corresponding irreducible unitary representation. So the final form of the partition function is

$$Z = \sum_{\lambda} d(\lambda)^{\chi(M)} \exp \left(-\frac{1}{\hbar} A_M C(\lambda) \right), \quad (3.26)$$

where λ denotes the irreducible unitary representation corresponding to the co-adjoint orbit, and $d(\lambda)$ is the dimension of this representation. This is exactly the partition function for the two-dimensional Yang-Mills theory [10]. When we omit the Casimir term in the action we get just a sum over the dimensions of the representations, which is the correct result for the BF-theory, see e.g. [19].

4 Concluding Remarks

The Poisson-Sigma model is more than a unified framework for different topological and semi-topological field theories. Due to its reformulation of the degrees of freedom of the theories in terms of the coordinates of a Poisson manifold it achieves a description in terms of the natural variables of general dynamical systems. Gauge theories, which are characterized by singular Lagrangians, cannot in general be described in terms of symplectic manifolds; the foliation which is characteristic for Poisson manifolds is necessary.

Such a description of gauge theories allows one to discuss the quantization by a direct application of the techniques of deformation quantization. The connection between the Poisson-Sigma model and the deformation quantization was shown by Cattaneo and Felder [6] by calculation of the perturbation expansion in the covariant gauge. Our nonperturbative calculation of the path integral which depends essentially on the framework of Poisson manifolds leads to an almost closed expression for the partition function. In the special case of a linear Poisson structure we were able to calculate the well known formula for the partition function of the SU(2) Yang-Mills theory with the help of fundamental facts of the representation theory of groups and algebras. We believe that further research will uncover ways of utilizing these structures more thoroughly. The techniques used here should in principle be applicable in more general situations than the particular case in Section (3.4). Finally, an understanding of the mechanisms active in the general case could help to understand the structure of gauge theories in a more fundamental way.

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References

- [1] P.SCHALLER, T.STROBL: *Poisson- σ -models: A Generalization of 2d Gravity-Yang-Mills Systems*, Talk delivered at the Conference on Integrable Systems, Dubna 1994, e-Print Archive: **hep-th/9411163**
- [2] N.IKEDA: *Two-dimensional Gravity and Nonlinear Gauge Theory*, Ann.Phys. **235** 1994, 435, e-Print Archive: **hep-th/9312059**
- [3] P.SCHALLER, T.STROBL: *Poisson Structure Induced (Topological) Field Theories in two Dimensions*, Mod. Phys. Lett. **A9** (1994), 3129

- [4] M.HENNEAUX, C.TEITELBOIM: *Quantization of Gauge Systems*, Princeton University Press, Princeton, New Jersey (1992)
- [5] I.A.BATALIN, G.A.VILKOVISKY: *Gauge Algebra and Quantization*, Phys. Lett. **69B** (1977), 309
- [6] A.S.CATTANEO, G.FELDER: *A Path Integral Approach to the Kontsevich Quantization Formula*, e-Print Archive: **math/9902090**
- [7] M.KONTSEVICH: *Deformation Quantization of Poisson Manifolds I*, e-Print Archive: **q-alg/9709040**
- [8] W.KUMMER, H.LIEBL, D.V.VASSILEVICH: *Exact Path Integral Quantization of generic 2d Dilaton Gravity*, Nucl.Phys **B493** (1997), 491
- [9] A.C.HIRSHFELD, T.SCHWARZWELLER: *Path Integral Quantization of the Poisson-Sigma Model*, e-Print Archive: **hep-th/9910178** (to appear in Ann.Phys. (Leipzig))
- [10] A. MIGDAL: *Recursion Relations in Gauge Theories*, Zh. Eksper. Teoret. Fiz. **69** (1975), 810 (Soviet Physics JETP. **42**, 413).
E.WITTEN: *On Quantum Gauge Theories in Two Dimensions*, Comm. Math. Phys. **141** (1991), 153
M.BLAU, G.THOMPSON: *Quantum Yang-Mills Theory on Arbitrary Surfaces*, Int. J. Mod. Phys. **A7** (1992), 3781
- [11] I.VAISMAN: *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics Volume **118**, Birkhäuser, Basel, 1994
- [12] A.WEINSTEIN: *The Local Structure of Poisson Manifolds*, J. Differential Geometry **18** (1983), 523
- [13] E.WITTEN: *Topological Sigma Models*, Comm.Math.Phys. **118** (1988), 411
- [14] L.BAULIEU, I.SINGER: *The topological Sigma Model*, Comm.Math.Phys. **125** (1989), 227
- [15] D.BIRMINGHAM, M.BLAU, G.THOMPSON: *Topological Field Theory*, Phys.Rept. **209** (1991), 129
- [16] W.KUMMER, D.J.SCHWARZ: *General analytic Solution of R^2 Gravity with dynamical Torsion in two Dimensions*, Phys.Rev. **D 45** (1992), 3628
- [17] J.GOMIS, J.PARIS, S.STUART: *Antibrackets, Antifields and Gauge-Theory Quantization*, Phys. Rept. **259** (1995), 1
- [18] A.KIRILLOV: *The Orbit Method I, Geometric Quantization*, in Representation Theory of Groups and Algebras, Contemporary Mathematics Volume **145**, 1993
- [19] M.BLAU, G.THOMPSON: *Lectures on 2d Gauge Theories*, presented at the 1993 Trieste Summer School in High Energy Physics and Cosmology, e-Print Archive: **hep-th/9310144**
- [20] A.KIRILLOV: *Elements of the Theory of Representations*, Grundlehren der mathematischen Wissenschaften **220**, Springer, Berlin, 1976